# Assignment 4

1. What is the most efficient means of evaluating the polynomial $x^5 + 3x^2 + 4x + 2$?

Horner's rule: $((x^3 + 3)x + 4)x + 2$

2. You are introduced to a programming language which has the following features:

   a. assignment is performed with the `:=` operator,
   b. the operators for addition and multiplication are `+` and `*`, respectively,
   c. the language is untyped, so you do not have to declare the type of a local variable, you simply start using it,
   d. a for loop is constructed as follows:
   ```
   for i from m to n do
        # for loop body
   end do
   ```
   e. an array may be indexed from arbitrary initial and final values, including negative indices, and arrays are indexed, like in C++, using square brackets.

Suppose that the coefficients of a polynomial are stored in an array indexed from `0` to `n` and the name of the array is `coeffs`.

3. Evaluate the polynomials

```
-0.2916666666666667*x^4 + 17737.75*x^3 - 404521422.2083333*x^2

                        + 4100171288508.750*x – 15584531083653020.0
```

and

```
-0.2916666666666667*x^4 - 0.25*x^3 + 2.791666666666667*x^2

                        + 1.75*x + 4.0
```

at the values $x = 15203.6$ and $x = –0.4$, respectively. The first is a polynomial interpolating the points

(15202, 9), (15203, 5), (15204, 4), (15205, 8), (15206, 12)

and the second interpolates

(–2, 9), (–1, 5), (0, 4), (1, 8), (2, 12).

Should these two evaluate the same result? Which calculation is likely more accurate?

These should evaluate to the same result, but even using Horner's rule, the first evaluates to 4.0, while the second evaluates to 3.7552, and the second is the exact answer.

4. Explain how you would go around finding and evaluating the polynomial that interpolates the four points (15.2, 0.5), (15.3, 0.7), (15.4, 0.4), (15.5, 0.2) at the point $x = 15.32$? What would be the Vandermonde matrix? What is the system of linear equations you are solving? At what point do you evaluate the interpolating polynomial at? You do not have to solve any system of linear equations, but you can indicate what you would do at each step under the assumption you have performed the previous calculation.

We want to evaluate a point on the middle interval (15.3, 15.4), and thus, we would shift and scale the mid-point 15.35 to the origin so that the points are (–1.5, 0.5), (–0.5, 0.7), (0.5, 0.4) and (1.5, 0.2), and then evaluate this polynomial at (15.32 – 15.35)/0.1 which equals –0.3. The 0.1 is the step size between the four $x$ values. This would have the Vandermonde matrix $\begin{pmatrix} -3.375 & 2.25 & -1.5 & 1 \\ -0.125 & 0.25 & -0.5 & 1 \\ 0.125 & 0.25 & 0.5 & 1 \\ 3.375 & 2.25 & 1.5 & 1 \end{pmatrix}$, the **y** vector is $\begin{pmatrix} 0.5 \\ 0.7 \\ 0.4 \\ 0.2 \end{pmatrix}$, and the solution is $\begin{pmatrix} 0.1 \\ -0.1 \\ -0.325 \\ 0.757 \end{pmatrix}$, and thus the polynomial is $((0.1x - 0.1)x - 0.325)x + 0.757$ and because the mid-point is 15.35, evaluating this polynomial at $\delta = (15.32 - 15.35)/0.1 = -0.3$ yields 0.6608.

5. Explain how you would go around finding and evaluating the polynomial that interpolates the three points (15.2, 0.5), (15.3, 0.7), (15.4, 0.4) at the point $x = 15.38$ assuming you do not have access to information at $x = 15.5$. What would be the Vandermonde matrix? What is the system of linear equations you are solving? At what point do you evaluate the interpolating polynomial at? You do not have to solve any system of linear equations, but you can indicate what you would do at each step under the assumption you have performed the previous calculation.

You would shift and scale the mid-point 15.35 so that the points are now (–1.5, 0.5), (–0.5, 0.7) and (0.5, 0.4) and then evaluate this at (15.38 – 15.35)/0.1 = 0.3 where 0.1 is the step size between the four points. The Vandermonde matrix is $\begin{pmatrix} 2.25 & -1.5 & 1 \\ 0.25 & -0.5 & 1 \\ 0.25 & 0.5 & 1 \end{pmatrix}$ and the solution is $\begin{pmatrix} -0.25 \\ -0.3 \\ 0.6125 \end{pmatrix}$ and thus the polynomial is $(-0.25x - 0.3)x + 0.6125$. Evaluated at $x = 0.3$ yields 0.5.

6. Approximate the derivative and second derivative at the point $x = 3.2$ given the three points (3.1, 4.7), (3.2, 4.9) and (3.3, 5.0).

$(5.0 - 4.7)/0.2 = 1.5$ and $(5.0 - 2\cdot 4.9 + 4.7)/0.1^2 = -10$

7. Approximate the derivative and second derivative at the point $x = 3.3$ given the four points (3.0, 4.4), (3.1, 4.7), (3.2, 4.9) and (3.3, 5.0) so that the result is $O(h^2)$.

$(3\cdot 5.0 - 4\cdot 4.9 + 4.7)/0.2 = 0.5$ and $(2\cdot 5.0 - 5\cdot 4.9 + 4\cdot 4.7 - 4.4)/0.1^2 = -10$

8. Calculate the second derivative of $\sin(x)$ at the point $x = 7.3$ using the points $x = 7.1$, 7.2, and 7.3, and then again using the points $x = 7.2$, 7.3, and 7.4. Which has the larger error, and why?

$(\sin(7.1) - 2\sin(7.2) + \sin(7.3))/0.1^2 = -0.79300669438653$

$(\sin(7.2) - 2\sin(7.3) + \sin(7.4))/0.1^2 = -0.84972815963491$

The second is the centered divided-difference formula that is $O(h^2)$ but is also centered, and thus likely more correct.

The first is a backward divided-difference formula that is O($h$) but as it is not centered, it is also less likely to be correct.

This is reflected by the correct answer being –sin(7.3) = –0.8504366206285645, which is correct to approximately three significant digits.

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